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Maximum Likelihood Estimation Based on Step Stress-Partially Accelerated Life Testing for Topp Leone-Inverted Kumaraswamy Distribution

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ABSTRACT
Partially accelerated life tests are very important in life testing experiments because it saves the testing time a lot of manpower, material sources and money. Partially accelerated life tests are used when the data obtained from accelerated life tests cannot be extrapolated to usual use conditions. In this paper, step stress-partially accelerated life test is discussed based on Type II censored samples when the lifetime of items under usual use condition has Topp Leone-inverted Kumaraswamy distribution. The maximum likelihood estimators for the unknown parameters and the acceleration factor are obtained. Numerical study and some interesting comparisons are presented to illustrate the theoretical results. Also, two real data sets are applied to confirm the applicability in real life.

KEYWORDS
Topp Leone-inverted Kumaraswamy distribution; censored samples; asymptotic Fisher information matrix; step stress-partially accelerated life test.

1. Introduction

Today's increasing market competition and higher customer expectations are driving manufacturers to design and produce highly reliable products. It is important to assess and estimate the reliability of a product during the design and development stage because the time-to-market is getting shorter and shorter. Also, manufacturing designs are improving continuously due to advancement in technology; therefore, it is becoming more and more difficult to obtain information about lifetime of products or materials with high reliability at the time of testing under usual conditions. In such problems, *accelerated life testing* (ALT) or *partially ALT* (PALT) are preferred to be used in manufacturing industries to obtain enough failure data in a short period of time and necessary to study its relationship with external stress variables. Such testing could save much time, manpower, material sources and money.

The major assumption in ALT is that the mathematical model relating the lifetime of the unit and the stress are known or can be assumed. In some cases, such life stress relationships are not known and cannot be assumed i.e., ALT data cannot be extrapolated to usual use condition. So, in such cases, PALT is a more suitable test

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to be performed for which tested units are subjected to both usual and accelerated conditions. Chernoff (1962) and Bessler *et al.* (1962) introduced and studied the concept of ALT. Stress can be applied in different ways as constant stress, step stress and progressive stress among others.

In step stress-ALT loading, a specimen is subjected to successively higher levels of stress. A specimen is first subjected to a specified constant stress for a specified length of time. If it does not fail, it is subjected to a higher stress level for a specified time. The stress on a specimen is thus increased step by step until it fails. Usually, all specimens go through the same specified pattern of stress level saved test times. For more details about step stress-ALT [see Jian-Ping and Xin-Min (2005), Wang (2006), Balakrishnan and Han (2008), Wang (2010), Rezk *et al.* (2014), Hakamipour and Rezaei (2017) and Mohie El-Din *et al.* (2021)].

In a *step stress-PALT* (SS-PALT) a tested unit is first run at usual use condition and, if it does not fail for a specified time or number of failures, then it is run at accelerated use condition until failure occurs or the observation is censored. The objective of such experiment is to collect more failure data in a limited time without necessarily using a high stress to all test units. Some references in the field of Bayesian and non-Bayesian estimation based on SS-PALT under Type I, Type II and Type II progressive censoring include Abdel-Hamid and AL-Hussaini (2008), Ismail and Aly (2009), Ismail (2011), Shi *et al.* (2016), El-Dessouky (2017), Mahmoud *et al.* (2018), Aljohani and Alfar (2020).

This paper is organized as follows: in Section 2, a description of the model, the basic assumptions and test procedure are presented. The *maximum likelihood* (ML) estimators (point and interval) for the parameters and the acceleration factor for SS-PALT based on Type II censoring are derived in Section 3. Numerical illustration is given in Section 4. Finally, some general conclusions are introduced in Section 5.

2. Model Description and Basic Assumptions

In this section, the model description is presented in Subsection 2.1, the basic assumptions and test procedure are given in Subsection 2.2.

2.1. Model description

Behairy *et al.* (2020) introduced *Topp Leone-inverted Kumaraswamy* (TL-IK) distribution as a composite distribution of the TL(θ) and IK(a, b) distributions. It is denoted by TL-IK (a, b, θ), its *cumulative distribution function* (cdf) and *probability density function* (pdf) are given, respectively, by

$$F\left(x;\underline{\vartheta}\right)=\left[\varphi(a)\right]^{\theta b}\Big[2-\left(\varphi(a)\right)^{b}\Big]^{\theta}, \qquad 0<x<\infty;\quad \left(\underline{\vartheta}>\underline{0}\right), \tag{1}$$

where

$$\varphi\left(a\right)=1-\left(1+x\right)^{-a}, \underline{\vartheta}=\ (a,b,\theta)', \tag{2}$$

the pdf corresponding to (1) is given by

$$f\left(x;\underline{\vartheta}\right)=2ab\theta(1+x)^{-(a+1)}[\varphi(a)]^{\theta b-1}\left[1-(\varphi(a))^{b}\right]\left[2-(\varphi(a))^{b}\right]^{\theta-1},\quad 0<x<\infty;\left(\underline{\vartheta}>\underline{0}\right), \tag{3}$$

where a, b and θ are shape parameters.
The *reliability function* (rf) and *hazard rate function* (hrf) are given, respectively, by:

$$R\left(x_0;\underline{\vartheta}\right)= P\left(X\geq x_0\right)=1-\left[\varphi\left(a\right)\right]^{\theta b}\left[2-\left(\varphi\left(a\right)\right)^{b}\right]^{\theta},\quad 0<x_0<\infty;\quad\left(\underline{\vartheta}>\underline{0}\right), \tag{4}$$

and

$$h\left(x_0;\underline{\vartheta}\right)=\frac{f(x_0)}{1-F(x_0)}=\frac{2ab\theta(1+x_0)^{-(a+1)}[\varphi(a)]^{\theta b-1}\left[1-(\varphi(a))^{b}\right]\left[2-(\varphi(a))^{b}\right]^{\theta-1}}{1-[\varphi(a)]^{\theta b}\left[2-(\varphi(a))^{b}\right]^{\theta}},\quad 0<x_0<\infty;\quad\left(\underline{\vartheta}>\underline{0}\right). \tag{5}$$

Behairy *et al.* (2020) showed that the TL-IK (a, b, θ) distribution contains some special well-known distributions in lifetime, such as the TL-Lomax (Pareto Type II), the TL-log-logistic (Fisk), the Lomax and the log-logistic (Fisk) distributions. They derived some transformed distributions such as the TL-exponentiated Weibull, TL-exponentiated Burr Type XII, TL-Kumaraswamy Dagum and TL-Kumaraswamy-inverse Weibull, among several others. They studied the properties of this distribution, which include the stress-strength reliability, moments, moment generating and quantile functions of the TL-IK distribution. Also, they derived the ML estimators, *asymptotic variances and covariance matrix* (AVCM) of the ML estimators and *asymptotic confidence intervals* (ACIs) for the parameters, also they obtained the ML two-sample predictor for the future observation based on Type-II censored data. Behairy *et al.* (2019) introduced ML estimators and ML prediction of constant stress-PALT based on Type II censored sampling from TL-IK distribution. Also, AL-Dayian *et al.* (2021) presented Bayesian estimation and prediction of constant stress-PALT based on Type II censored sampling from TL-IK distribution.

2.2. Basic assumptions and test procedure

- **Basic assumptions**

  ▷ X is the lifetime of an item at usual condition follows the TL-IK distribution.

  ▷ The failure times Y_{ij} ; $i = 1, 2$; $j = 1, 2, \ldots, n_i$ are independent and identically distributed (i.i.d) random variables

  ▷ Two stress levels z_1 and z_2 (usual and high) are used.

  ▷ The total lifetime of test items denoted by Y passes through two stages, which are the usual and accelerated conditions. Then, the lifetime of an item under SS-PALT is

$$Y=\begin{cases} X & \text{if}\quad X\leq y_{1n_1} \\ y_{1n_1}+\beta^{-1}\left(X-y_{1n_1}\right) & \text{if } X>y_{1n_1} \end{cases}, \tag{6}$$

where, y_{1n_1} is the lifetime of the failure item order ${n_1}^{th}$, which is the latest failure under usual condition, and X denotes the lifetime under the stress level z_1.

▷ For any level of stress, the lifetime of test unit follows TL-IK distribution.

Test procedure

1. Suppose that n test units are initially placed on normal stress z_1 and run until time $y_{1n_1}$when exactly n_1 failures are observed while testing at the stress level z_1. If the number of failures reach n_1 items which is pre-specified the test is terminated, where $n_1 = \pi_1 n$ and π_1 is a proportion of total test items put on test under usual condition that is pre-specified. $0 < \pi_1 < 1$.
2. For items do not fail at usual use condition $(n-n_1)$ accelerated use condition and the $(n-n_1)$ units are put on high stress z_2 and run until time y_{2n_2} when exactly n_2 failures are observed. Where $n_2 = \pi_2 n$ and π_2 is a proportion of total test items put on test under accelerated condition that is pre-specified. $0 < \pi_2 < 1$ and $0 < \pi_1 + \pi_2 < 1$. The remaining $n_c = n - n_1 - n_2$ units are then censored.

In this case, if the item has not failed by some pre-specified number of failures, the test condition is switched to a higher level of stress and it is continued until another specified number of failures occur or the observations is censored. The effect of this switch is to multiply the remaining lifetime of the item by the inverse of an acceleration factor β, where $Y = \beta^{-1}X$, β is the acceleration factor which is the ratio of mean life at usual condition to that at accelerated condition and $\beta > 1$. Thus, the total lifetime of a test item, denoted by Y , passes through two stages, the first stage is the usual use condition and the second stage is the accelerated use condition, respectively. [See Ismail and Aly (2009) and El-Dessouky (2017)].

Applying simple SS-PALT under Type II censoring, then the pdf of total lifetime Y of an item is given by:

$$Y = \begin{cases} f_1\left(y;\underline{\vartheta}\right) & \text{if } \ y \leq y_{1n_1} \\ f_2\left(y;\underline{\Phi}\right) & \text{if } \ y > y_{1n_1} \end{cases}, \tag{7}$$

where $f_1\left(y\right)$ is given in (7) and has TL-IK distribution with the pdf

$$\begin{aligned} f_1\left(y\right) = 2ab\theta(1+y_{1j})^{-(a+1)}[\varphi_1(a)]^{\theta b-1}\left\{1-[\varphi_1(a)]^b\right\}\left\{2-[\varphi_1(a)]^b\right\}^{\theta-1}, \\ y \leq y_{1n_1};\ \left(\underline{\vartheta} > \underline{0}\right), \end{aligned} \tag{8}$$

$f_2\left(y\right)$ in (7) is obtained by the transformation-variable technique using $f_1\left(y\right)$ and the model is given by (3), with the pdf

$$\begin{aligned} f_2\left(y;\underline{\Phi}\right) = 2ab\theta\beta(D)^{-(a+1)}[\varphi_1(a\beta)]^{\theta b-1}\left[1-(\varphi_1(a\beta))^b\right]\left[2-(\varphi_1(a\beta))^b\right]^{\theta-1}, \\ y > y_{1n_1}; \left(\underline{\vartheta} > \underline{0}\right);\ \beta > 1, \end{aligned} \tag{9}$$

$$\text{where} \quad \begin{cases} \underline{\Phi} = \ (a,b,\theta,\beta)', \varphi_1\left(a\right) = 1-(1+y_{1j})^{-a},\ \varphi_1\left(a\beta\right) = 1-(D)^{-a}, \\ D = 1+\beta\left(y_{2j}-y_{1n1}\right)+y_{1n1}, \end{cases} \tag{10}$$

the cdf, rf and hrf of the pdf, $f_2(y;\underline{\Phi})$ for an item tested at acceleration conditions are given respectively, by

$$F(y;\underline{\Phi}) = [\varphi_c(a\beta)]^{\theta b}\left[2-(\varphi_c(a\beta))^b\right]^{\theta}, \qquad y > y_{1n_1}; (\underline{\vartheta} > \underline{0})\,;\ \beta > 1, \tag{11}$$

$$R(y_0;\underline{\Phi}) = \ 1-[\varphi_c(a\beta)]^{\theta b}\left[2-(\varphi_c(a\beta))^b\right]^{\theta}, \quad y > y_{1n_1}; (\underline{\vartheta} > \underline{0})\,;\ \beta > 1, \tag{12}$$

and

$$h(y_0;\underline{\Phi}) = \frac{f_2(y_0;\underline{\Phi})}{R(y_0;\underline{\Phi})} = \frac{2ab\theta\beta(D)^{-(a+1)}[\varphi_1(a\beta)]^{\theta b-1}\left[1-(\varphi_1(a\beta))^b\right]\left[2-(\varphi_1(a\beta))^b\right]^{\theta-1}}{1-[\varphi_c(a\beta)]^{\theta b}\left[2-(\varphi_c(a\beta))^b\right]^{\theta}}, \quad y > y_{1n_1}; (\underline{\vartheta} > \underline{0})\,;\ \beta > 1, \tag{13}$$

$$\varphi_c(a\beta) = 1-(D_1)^{-a}, D_1 = 1+\beta\,(y_{2n2}-y_{1n1})+y_{1n1}. \tag{14}$$

The observed values of the total lifetime Y are given by

$$y_{(1)} \le y_{(2)} \le \cdots \le y_{(n_1)} \le y_{(n_1+1)} \le \cdots \le y_{(n_1+n_2-1)} \le y_r,$$

where y_r is the lifetime of the $(n_1+n_2)th$ unit of the last failure unit in the experiment.

3. Maximum Likelihood Estimation

The ML estimation for the unknown parameters and the acceleration factor of the TL-IK distribution for SS-PALT under Type II censored data are discussed in Subsection 3.1. In Subsection 3.2, Cls for the parameters and the acceleration factor of the TL-IK distribution for SS-PALT under Type II censored data are obtained.

The *likelihood function* (LF) for the SS-PALT based on Type II censoring in its general form and n_c censored data is

$$\begin{aligned} L\left(\underline{\Phi};\ \underline{y}\right) \propto & \prod_{j=1}^{n_1} ab\theta(1+y_{1j})^{-(a+1)}[\varphi_1(a)]^{\theta b-1}\left\{1-[\varphi_1(a)]^b\right\}\left\{2-[\varphi_1(a)]^b\right\}^{\theta-1} \\ & \times \prod_{j=1}^{n_2} ab\theta\beta(D)^{-(a+1)}[\varphi_1(a\beta)]^{\theta b-1}\left\{1-[\varphi_1(a\beta)]^b\right\}\left\{2-[\varphi_1(a\beta)]^b\right\}^{\theta-1} \\ & \times \prod_{j=1}^{n_c}\left\{1-[\varphi_c(a\beta)]^{\theta b}\left[2-(\varphi_c(a\beta))^b\right]^{\theta}\right\}, \end{aligned} \tag{15}$$

where $\varphi_1(a)\,,\ \varphi_1(a\beta)\,, D, \varphi_c(a\beta)$ and $D_1$are given by (10) and (14) respectively.

3.1. Point estimation

The ML estimators of a, b, θ and β are obtained by maximizing the natural logarithm of (15), denoted by ℓ which can be written in the form:

$$\begin{aligned}
\ell \equiv lnL\left(\underline{\Phi};\ \underline{y}\right) &\propto n\pi_1 ln\,(a) + n\pi_1 \ln\,(b)\ +\ n\pi_1 \ln\,(\theta)\ -(a+1)\sum_{j=1}^{n_1} \ln\,(1+y_{1j}) \\
&+(\theta b-1)\sum_{j=1}^{n_1} \ln\left[\varphi_1\,(a)\right]\ +\sum_{j=1}^{n_1} ln\left\{1-[\varphi_1(a)]^b\right\} + (\theta-1)\sum_{j=1}^{n_1} ln\left\{2-[\varphi_1(a)]^b\right\} \\
&+n\pi_2 ln\,(a) + n\pi_2 \ln\,(b)\ +\ n\pi_2 \ln\,(\theta)\ + n\pi_2 ln\,(\beta) - (a+1)\sum_{j=1}^{n_2} lnD + (\theta b-1) \\
&\times \sum_{j=1}^{n_2} \ln\varphi_1\,(a\beta)\ +\sum_{j=1}^{n_2} ln\left\{1-[\varphi_1(a\beta)]^b\right\}\ +(\theta-1)\sum_{j=1}^{n_2} ln\left\{2-[\varphi_1(a\beta)]^b\right\} \\
&+n_c \ln\left\{1-[\varphi_c(a\beta)]^{\theta b}\left\{2-\varphi_c(a\beta)^b\right\}^{\theta}\right\}.
\end{aligned} \tag{16}$$

The partial derivatives of the logarithm of the LF with respect to a, b, θ and β are given below:

$$\begin{aligned}
\frac{\partial \ell}{\partial a} &= \frac{n(\pi_1+\pi_2)}{a} - \sum_{j=1}^{n_1} \ln\,(1+y_{1j})\ +(\theta b-1)\ \sum_{j=1}^{n_1} \frac{(1+y_{1j})^{-a} \ln\,(1+y_{1j})}{\varphi_1\,(a)} \\
&- \sum_{j=1}^{n_1} \frac{b(\varphi_1\,(a))^{b-1}\,(1+y_{1j})^{-a} \ln\,(1+y_{1j})}{\left\{1-[\varphi_1(a)]^b\right\}} - (\theta-1)\sum_{j=1}^{n_1} \frac{b(\varphi_1\,(a))^{b-1}\,(1+y_{1j})^{-a} \ln\,(1+y_{1j})}{\left\{2-[\varphi_1(a)]^b\right\}} \\
&- \sum_{j=1}^{n_2} lnD + (\theta b-1)\sum_{j=1}^{n_2} \frac{(D)^{-a} \ln\,(D)}{\varphi_1\,(a\beta)}\ - \sum_{j=1}^{n_2} \frac{b(\varphi_1\,(a\beta))^{b-1}\,(D)^{-a} \ln\,(D)}{\left\{1-[\varphi_1(a\beta)]^b\right\}} -\ (\theta-1) \\
&\times \sum_{j=1}^{n_2} \frac{b(\varphi_1\,(a\beta))^{b-1}\,(D)^{-a} \ln\,(D)}{\left\{2-[\varphi_1(a\beta)]^b\right\}} \\
&- n_c\left[\frac{\theta b\,(D_1)^{-a} ln\,(D_1)\ \{(2-[\varphi_c(a\beta)]^b)^{\theta}[\varphi_c(a\beta)]^{\theta b-1} -\ [\varphi_c(a\beta)]^{b(\theta+1)-1}\left\{2-[\varphi_c(a\beta)]^b\right\}^{\theta-1}\}}{\left\{1-[\varphi_c(a\beta)]^{\theta b}\left\{2-[\varphi_c(a\beta)]^b\right\}^{\theta}\right\}}\right],
\end{aligned} \tag{17}$$

$$\frac{\partial \ell}{\partial b}=\frac{n(\pi_1+\pi_2)}{b}+\theta\sum_{j=1}^{n_1} ln\varphi_1(a)\ -\sum_{j=1}^{n_1}\frac{(\varphi_1(a))^b \ln(\varphi_1(a))}{\left\{1-[\varphi_1(a)]^b\right\}}-(\theta-1)\sum_{j=1}^{n_1}\frac{(\varphi_1(a))^b \ln(\varphi_1(a))}{\left\{2-[\varphi_1(a)]^b\right\}}$$
$$+\ \theta\sum_{j=1}^{n_2} ln\varphi_1(a\beta)\ -\sum_{j=1}^{n_2}\frac{(\varphi_1(a\beta))^b ln(\varphi_1(a\beta))}{\left\{1-[\varphi_1(a\beta)]^b\right\}}-(\theta-1)\sum_{j=1}^{n_2}\frac{(\varphi_1(a\beta))^b ln(\varphi_1(a\beta))}{\left\{2-[\varphi_1(a\beta)]^b\right\}}$$
$$-\,n_c\left[\frac{\left\{2-[\varphi_c(a\beta)]^b\right\}^{\theta}\theta\,[\varphi_c(a\beta)]^{\theta b} ln(\varphi_c(a\beta))\ -[\varphi_c(a\beta)]^{b(\theta+1)}\theta\left\{2-[\varphi_c(a\beta)]^b\right\}^{\theta-1} ln\,[\varphi_c(a\beta)]}{\left\{1-[\varphi_c(a\beta)]^{\theta b}\left\{2-[\varphi_c(a\beta)]^b\right\}^{\theta}\right\}}\right], \tag{18}$$

$$\frac{\partial \ell}{\partial \theta}=\frac{n(\pi_1+\pi_2)}{\theta}+b\sum_{j=1}^{n_1} ln\varphi_1(a)\ +\sum_{j=1}^{n_1}\left\{2-[\varphi_1(a)]^b\right\}$$
$$+\ b\sum_{j=1}^{n_2} ln\varphi_1(a\beta)\ +\sum_{j=1}^{n_2}\left\{2-[\varphi_1(a\beta)]^b\right\}$$
$$-\,n_c\left[\frac{\left\{2-[\varphi_c(a\beta)]^b\right\}^{\theta} b\,[\varphi_c(a\beta)]^{\theta b} ln(\varphi_c(a\beta))\ +[\varphi_c(a\beta)]^{\theta b}\left\{2-[\varphi_c(a\beta)]^b\right\}^{\theta} ln\left\{2-[\varphi_c(a\beta)]^b\right\}}{\left\{1-[\varphi_c(a\beta)]^{\theta b}\left\{2-[\varphi_c(a\beta)]^b\right\}^{\theta}\right\}}\right], \tag{19}$$

and the first partial derivative of the acceleration factor is

$$\frac{\partial \ell}{\partial \beta}=\frac{n\pi_2}{\beta}-(a+1)\sum_{j=1}^{n_2}\frac{(y_{2j}-y_{1n_1})}{D}+(\theta b-1)\sum_{j=1}^{n_2}\frac{a\,(y_{2j}-y_{1n_1})\,[\varphi_1(a\beta)]^{b-1}(D)^{-(a+1)}}{\varphi_1(a\beta)}$$
$$-\sum_{j=1}^{n_2}\frac{ab\,(y_{2j}-y_{1n_1})\,[\varphi_1(a\beta)]^{b-1}(D)^{-(a+1)}}{\left\{1-[\varphi_1(a\beta)]^b\right\}}-(\theta-1)\sum_{j=1}^{n_2}\frac{ab\,(y_{2j}-y_{1n_1})\,[\varphi_1(a\beta)]^{b-1}(D)^{-(a+1)}}{\left\{2-[\varphi_1(a\beta)]^b\right\}}$$
$$-\,n_c\left[\frac{ab\theta\,(y_{2n_2}-y_{1n_1})\,(D_1)^{-(a+1)}\{\ \left\{2-[\varphi_c(a\beta)]^b\right\}^{\theta}[\varphi_c(a\beta)]^{\theta b-1}-[\varphi_c(a\beta)]^{b(\theta+1)-1}\left\{2-[\varphi_c(a\beta)]^b\right\}^{\theta-1}\}}{\left\{1-[\varphi_c(a\beta)]^{\theta b}\left\{2-[\varphi_c(a\beta)]^b\right\}^{\theta}\right\}}\right], \tag{20}$$

where $\varphi_1(a)$, $\varphi_1(a\beta)$, $D, \varphi_c(a\beta)$ and $D_1$are given by (10) and (14) respectively. The ML estimators are obtained by setting (17)-(20) to zeros. The system of the non-linear equations can be solved numerically, to evaluate the ML estimates of $\hat{a}$, $\hat{b}$, $\widehat{\theta}$ and $\hat{\beta}$.

3.2. Asymptotic Confidence intervals

The AVCM of the estimators a, b, θ and β are derived depending on the inverse *asymptotic Fisher information matrix* (AFIM) using the second partial derivatives of the logarithm of the LF.
The AFIM can be written as follows:

$$\tilde{I} \approx -\left[\frac{\partial^2 l}{\partial\Phi_i\partial\Phi_j}\right], \qquad\qquad i,j=1,\ 2,\ 3,4, \tag{21}$$

where $\Phi_1 = a, \Phi_2 = b,\ \Phi_3 = \theta$ and $\Phi_4 = \beta$.
For large sample size, the ML estimators under regularity conditions are consistent and asymptotically unbiased as well as asymptotically normally distributed. Therefore, the ACIs for the parameters; Φ, can be obtained by
$\mathrm{P}\left[-\mathrm{Z} < \frac{\widehat{\Phi}_{\mathrm{iML}}-\Phi_{\mathrm{i}}}{\sigma_{\widehat{\Phi}_{\mathrm{iML}}}} < \mathrm{Z}\right] = 1-\tau$, where Z is the $100\left(1-\frac{\tau}{2}\right)$th standard normal percentile. The two-sided $100(1-\tau)\%$ ACIs are

$$\mathrm{L}_{\Phi_i}{=}\widehat{\Phi}_{\mathrm{iML}}{-}\mathrm{Z}_{\frac{\tau}{2}}\ \widehat{\sigma}_{\widehat{\Phi}_{\mathrm{iML}}},\ \text{and}\ \mathrm{U}_{\Phi_i}{=}\widehat{\Phi}_{\mathrm{iML}}{+}\mathrm{Z}_{\frac{\tau}{2}}\ \widehat{\sigma}_{\widehat{\Phi}_{\mathrm{iML}}}, \tag{22}$$

where $\widehat{\sigma}_{\widehat{\Phi}_{\mathrm{iML}}}$ is the standard deviation and $\widehat{\Phi}_{\mathrm{iML}}$ *is* $\hat{a}$, $\widehat{b,}$ $\widehat{\theta}$, or $\widehat{\beta}$, respectively.

4. Numerical Illustration

This section aims to investigate the precision of the theoretical results of estimation based on simulated and real data.

4.1. Simulation algorithm

In this subsection, a simulation study is conducted to illustrate the performance of the presented ML estimates based on generated data from the TL-IK $(a,\ b,\ \theta)$ distribution considering the SS-PALT. The ML averages of the parameters, based on Type II censoring are computed. Moreover, the ACIs of the parameters are calculated. Simulation study is performed using Mathematica 11 for illustrating the obtained results.
The steps of the simulation procedure based on Type II censored data are as follows:
Step 1: For given values of $\underline{\vartheta}$, random samples of size n are generated from the TL-IK $(a,\ b,\ \theta)$ distribution.

- The transformation between uniform distribution and TL-IK distribution is obtained as follows:

$$x = \left[\left(1-(u)^{\frac{1}{b}}\right)^{-\frac{1}{a}} - 1\right]\left(1-\sqrt{1-(u)^{\frac{1}{\theta}}}\right), \quad 0<u<1,$$

where $u_1, u_2, \ldots, u_n$ are random samples from uniform (0,1).
Step 2: The experiment is done under Type II censoring and r is the level of censoring and it can be determined as follows: $n_c = n{-}n_1{-}n_2$, which means that the experiment terminates when reaching the first number of failures. Each of the n test items are first run at usual condition, if the number of failures reach $n_1 = \pi_1 n$ items which is pre-specified, then the test is terminated, where $\pi_1 = 20\%$ is a proportion of total test items put on test under usual condition. Next the survival items ($80\%\ n$) are put on accelerated usual condition and run until the number of failures reach n_2 items, then the test is terminated, where $n_2 = \pi_2 n$ and $\pi_2 = 20\%$ is a proportion of total test items put on test under accelerated condition that is pre-specified.

Step 3: For each sample and for different combinations of the population parameter values, the distribution parameters and the acceleration factor are estimated in SS-PALT under Type II censored samples. Newton Raphson technique is applied for solving the nonlinear equations (17)-(20) to get the estimates of a, b, θ, and β.

- Repeat the preceding steps N times, where N denotes a predetermined number of simulated samples and N = 2000 is the number of repetitions.

Step 4: Evaluating the performance of the estimates is considered through some measurements of accuracy. In order to study the precision and variation of the estimates, it is convenient to use the average and the *estimated risk* (ER), where
$\overline{\widehat{\Phi}_i} = \frac{\sum_{j=1}^{N} \widehat{\Phi}_i^j}{N}$, $\Phi_i = a, b$, θ and β and ER= $\frac{\sum_{j=1}^{N} (estimate-true\ value)^2}{N}$.
Step 5: The two-sided asymptotic confidence limits with confidence levels of the acceleration factor and the three parameters are constructed using (22).

- Simulation results of the ML averages of the estimates are displayed in Tables 1-3, where the samples of size (n=30, 60, 100), are used. For each sample size, the chosen population parameters values are (Case 1, $a = 1.6$, $b = 1.5$, $\theta = 1.2$, $\beta = 1.4$), (Case 2, $a = 1.6$, $b = 1.5$, $\theta = 1.2$, $\beta = 2.4$) and (Case 3, $a = 1.6$, $b = 1.5$, $\theta = 1.2$, $\beta = 3.4$).
- Tables 1-3 present the ML averages, ERs and ACIs of the unknown parameters and the acceleration factor based on Type II censoring in Case 1, Case 2 and Case 3, respectively.

4.2. Applications

This subsection demonstrates how the proposed method can be used in practice through two real lifetime data sets. The TL-IK (a, b, θ) distribution is fitted to the two real data using Kolmogorov-Smirnov goodness of fit test via the R programming language.
Application 1
The data was given by Murthy *et al.* (2004), it refers to the time between failures for a repairable item: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86 and 1.17.
Application 2
The second application was provided by Dumonceaux and Antle (1973), where the data represents the maximum flood level (in millions of cubic feet per second) for the Susquehanna River at Harrisburg, Pennsylvania. Each number is the maximum flood level for a four-year period, the first, 0.654, being for the period 1890-1893, and the last, 0.265, being for the period 1966-1969. The data is
0.654, 0.613, 0.315, 0.449, 0.297, 0.402, 0.379, 0.423, 0.379, 0.3235 0.269, 0.740, 0.418, 0.412 0.494, 0.416, 0.338, 0.392, 0.484, 0.265.
The Kolmogorov–Smirnov goodness of fit test is applied on the two applications to check the validity of the fitted model. The p-values are given, respectively, by 0.5860 and 0.6465. The p-value given in each case showed that the model fits the data very well.

- Table 4 displays the ML estimates and *standard errors* (SE) of the unknown parameters and the acceleration factor, for the real data sets based on Type II censoring.

4.3. Concluding remarks

a) From Tables 1-3, one can observe that the ML averages are very close to the population parameter values as the sample size increases. Also, the ERs decrease when the sample size increases. This implies that the estimates are consistent and approach the true parameter values as the sample size increases.
b) The lengths of the ACIs of the parameters become narrower as the sample size increases.
c) Table 1 indicates that when the proportion π of the sample items allocated to the accelerated conditions decreases, the accuracy of the ERs gets better.
d) Also, from Tables 1-3 one can notice that, in most cases as the acceleration factor increases the ERs of $(a,\ b,\ \theta)$ decreases.
e) From Table 4, when the proportion π of the sample items allocated to the accelerated conditions decreases, the SE performs better.

5. General Conclusion

For products having high reliability, the test of product life under usual conditions often requires a long period of time. Therefore, ALT or PALT is used to facilitate estimating the reliability of the unit in a short period of time. In ALT test items are run only at accelerated conditions but in some cases, such relationship cannot be known or assumed. Thus, PALT is often used in such cases, where the test items are run at both usual and accelerated conditions. This paper deals with the SS-PALT under Type II censoring. It is assumed that the lifetime of test units has the TL-IK distribution. The ML estimators of the acceleration factor and the parameters are derived. From the results, one can conclude that, in most cases as the acceleration factor increases, the ERs of $(a,\ b,\ \theta)$ decrease. As the sample size increases the ERs and the length of the ACIs for the parameters and the acceleration factor decrease. This implies that the ML estimators of the parameters and the acceleration factor are asymptotically normally distributed and consistent.

References

[1] Abdel-Hamid, A. H. and AL-Hussaini, E. K. (2008). Step partially accelerated life tests under finite mixture models. *Journal of Statistical Computation and Simulation,* Vol. 78, No. 10, pp. 911-924.
[2] AL-Dayian, G. R., EL-Helbawy, A. A., Refaey, R. M. and Behairy, S. M. (2021). Bayesian estimation and prediction based on constant stress-partially accelerated life testing for Topp Leone-inverted Kumaraswamy distribution. *Journal of Advances in Mathematics and Computer Science*, Vol. 36, No. 3, pp. 11-32.
[3] Aljohani, H. M. and Alfar, N. M. (2020). Estimations with step-stress partially accelerated life tests for competing risks Burr XII lifetime model under Type-II censored data. *Alexandria Engineering Journal,* Vol. 59, No. 3, pp. 1171-1180.
[4] Balakrishnan, N. and Han, D. (2008). Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under Type II censoring. *Journal of Statistical Planning and Inference,* Vol. 138, No. 12, pp. 4172-4186.
[5] Behairy, S. M., Refaey, R. M., AL-Dayian, G. R. and EL-Helbawy, A. A. (2019). Estimation and prediction based on constant stress-partially accelerated life testing for Topp Leone-inverted Kumaraswamy distribution. This paper was presented and discussed in

"The 54^{th} Annual Conference on Statistics, Computer Sciences and Operation Research 9-11 Dec".

[6] Behariy, S. M., Refaey, R. M., EL-Helbawy, A. A. and AL-Dayian, G. R. (2020). Topp Leone-inverted Kumaraswamy distribution: properties, estimation and prediction. *Journal of Applied Probability and Statistics*, Vol. 15, No.3, pp. 93-118.

[7] Bessler, S., Chernoff, H. and Marshall, A. W. (1962). An optimal sequential accelerated life test. *Technometrics,* Vol. 4, No. 3, pp.367-379.

[8] Chernoff, H. (1962). Optimal accelerated life designs for estimation, *Technometrics,* Vol. 4 No. 3, pp. 381-408.

[9] Dumonceaux, R. and Antle, C., E. (1973). Discrimination between the log-normal and the Weibull distributions. *Technometrics,* Vol. 15, No. 4, pp. 923-926.

[10] El-Dessouky, E. A. (2017). Estimating costs of maintenance service policy using step stress partially accelerated life testing for the extension of exponential distribution under Type II censoring. *International Journal of Contemporary Mathematical Sciences,* Vol. 12, No. 5, pp.225-242.

[11] Hakamipour, N. and Rezaei, S. (2017). Optimizing the simple step stress accelerated life test with Type I censored Fréchet data. *REVSTAT – Statistical Journal*, Vol. 15, No. 1, pp, 1–23.

[12] Ismail, A. A. and Aly, H. M. (2009). Optimal planning of failure-step stress partially accelerated life tests under Type II censoring, *International Journal of Mathematical Analysis,* Vol. 3, No. 31, pp. 1509-1523.

[13] Ismail, A. (2011). Planning step-stress life tests with Type II censored data. *Scientific Research and Essays,* Vol. 6, No. 19, pp, 4021-4028.

[14] Jian-Ping, Z. and Xin-Min, G. (2005). Constant-step stress accelerated life test of VFD under Weibull distribution case. *Journal of Zhejiang University Science A*, Vol. 6, No. 7, pp. 722-727.

[15] Murthy, D. N. P., Xie, M. and Jiang, R. (2004). *Weibull Models.* John Wiley and Sons, Inc., Hoboken, New Jersey.

[16] Mohie El-Din, M. M., Abd El-Raheem, A. M. and Abd El-Azeem, S. O. (2021). On step-stress accelerated life testing for power generalized Weibull distribution under progressive Type-II censoring. *Annals of Data Science,* Vol. 8, No. 3, pp. 629-644. https://doi.org/10.1007/s40745-020-00270-4.

[17] Mahmoud, M. A. W., EL-Sagheer, R. M. and Abou-Senna, A. M. (2018). Estimating the modified Weibull parameters in presence of step-stress partially accelerated life testing. *Journal of Statistics Applications and Probability,* Vol. 7, No. 1, pp. 137-150.

[18] Rezk, H. R., EL-Helbawy, A. A. and AL-Dayian, G. R. (2014). Statistical inference for a simple step-stress model based on censored data from Kumaraswamy Weibull distribution. *British Journal of Mathematics and Computer Science*, Vol. 4, No. 11, pp. 1496-1514.

[19] Shi, X., Liu, F. and Shi, Y. (2016). Bayesian inference for step-stress partially accelerated competing failure model under Type II progressive censoring. *Mathematical Problems in Engineering,* Vol. 2016, pp. 1-10.

[20] Wang, B. (2006). Unbiased estimations for the exponential distribution based on step-stress accelerated life testing data. *Applied Mathematics and Computation,* Vol. 173, No. 2, pp. 1227-1237.

[21] Wang, B. X. (2010). Interval estimation for exponential progressive Type-II censored step-stress accelerated life-testing. *Journal of Statistical Planning and Inference*, Vol. 140, No. 9, pp. 2706-2718.

Appendix

Table 1: ML averages, estimated risks and 95% ACIs for the parameters a, b, θ, β based on Type II censoring,
(N=2000. $\pi_1 = 20\%$, $\pi_2 = 20\%, r = 60\%, \pi_1 = 20\%, \pi_2 = 50\%, r = 30\%, \pi_1 = 20\%, \pi_2 = 70\%, r = 10\%$)
(Case 1, $a = 1.6, b = 1.5, \theta = 1.2, \beta = 1.4$)

n	π	Parameters	Averages	ERs	UL	LL	Length
30	$\pi_1 = 20\%$	a	1.43995	0.00803	1.57039	1.30951	0.26088
		b	1.33229	0.08012	1.51243	1.15215	0.36028
		θ	11.04669	0.07236	1.22408	0.86930	0.35478
	$\pi_2 = 20\%$	β	1.38320	0.01334	1.60717	1.15922	0.44795
	$\pi_1 = 20\%$	a	1.29806	0.07708	1.67148	0.92464	0.74684
		b	1.18354	0.22381	1.62342	0.74366	0.87976
		θ	0.90827	0.19858	1.32465	0.49189	0.83276
	$\pi_2 = 50\%$	β	1.23379	0.03650	1.4185	1.04911	0.36937
	$\pi_1 = 20\%$	a	1.21120	0.12749	1.62272	1.62272	0.82304
		b	1.03732	0.39003	1.56844	0.50621	1.06223
		θ	0.77161	0.34442	1.27217	0.27105	1.00112
	$\pi_2 = 70\%$	β	1.15834	0.07537	1.41366	0.90303	0.51064
60	$\pi_1 = 20\%$	a	1.45043	0.00445	1.53786	1.36299	0.17486
		b	1.34831	0.06639	1.45635	1.24027	0.21607
		θ	1.06257	0.05913	1.16555	0.95959	0.20596
	$\pi_2 = 20\%$	β	1.37372	0.003112	1.47018	1.27727	0.19291
	$\pi_1 = 20\%$	a	1.31846	0.05898	1.63467	1.00225	0.63242
		b	1.21644	0.18014	1.57258	0.86029	0.71229
		θ	0.94056	0.15828	1.27485	0.60627	0.66858
	$\pi_2 = 50\%$	β	1.24022	0.02985	1.36898	1.11146	0.25752
	$\pi_1 = 20\%$	a	1.20272	0.12621	1.58393	0.82150	0.76243
		b	1.04356	0.36857	1.51942	0.56769	0.95172
		θ	0.77941	0.32288	1.22578	0.33304	0.89275
	$\pi_2 = 70\%$	β	1.15543	0.07016	1.35475	0.95611	0.39863
100	$\pi_1 = 20\%$	a	1.45890	0.00264	1.51946	1.28836	0.12112
		b	1.35843	0.05964	1.42849	1.39833	0.14014
		θ	1.07216	0.05304	1.13806	1.00627	0.13179
	$\pi_2 = 20\%$	β	1.36967	0.00182	1.42859	1.31075	0.11784
	$\pi_1 = 20\%$	a	1.33053	0.04269	1.56226	1.09880	0.46346
		b	1.23349	0.15133	1.48903	0.97795	0.51109
		θ	0.95722	0.13232	1.19581	0.71862	0.47719
	$\pi_2 = 50\%$	β	1.24221	0.02735	1.33936	1.14506	0.19429
	$\pi_1 = 20\%$	a	1.20834	0.11544	1.54989	0.86678	0.68311
		b	1.05523	0.34352	1.47902	0.63145	0.84758
		θ	0.79128	0.29965	1.18742	0.39513	0.79229
	$\pi_2 = 70\%$	β	1.15422	0.06922	1.33815	0.97028	0.36788

Table 2: ML averages, estimated risks and 95% ACIs for the parameters a, b, θ, β based on Type II censoring, (N=2000, $\pi_1 = 20\%, \pi_2 = 20\%, r = 60\%$)
(Case 2, $a = 1.6, b = 1.5, \theta = 1.2, \beta = 2.4$)

n	π	Parameters	Averages	ERs	UL	LL	Length
		a	1.44052	0.00852	1.57881	1.30223	0.27658
	$\pi_1 = 20\%$	b	1.33315	0.08050	1.52206	1.14423	0.37784
30		θ	1.04771	0.07242	1.23118	0.86423	0.36695
	$\pi_2 = 20\%$	β	2.32464	0.02389	2.58913	2.06015	0.52898
		a	1.45550	0.00347	1.53113	1.37987	0.15126
	$\pi_1 = 20\%$	b	1.35441	0.06259	1.44805	1.26076	0.18729
60		θ	1.06837	0.05572	1.15745	0.97929	0.17816
	$\pi_2 = 20\%$	β	2.31195	0.01093	2.42250	2.20140	0.22110
		a	1.46341	0.00206	1.51606	1.41077	0.10529
	$\pi_1 = 20\%$	b	1.36363	0.05686	1.42513	1.30213	0.12299
100		θ	1.07702	0.05059	1.13490	1.01914	0.11576
	$\pi_2 = 20\%$	β	2.31077	0.00945	2.38637	2.23516	0.15121

Table 3: ML averages, estimated risks and 95% ACIs for the parameters a, b, θ, β based on Type II censoring, (N=2000, $\pi_1 = 20\%, \pi_2 = 20\%, r = 60\%$)
(Case 3, $a = 1.6, b = 1.5, \theta = 1.2, \beta = 3.4$)

n	π	Parameters	Averages	ERs	UL	LL	Length
		a	1.44488	0.00739	1.57430	1.31545	0.25885
	$\pi_1 = 20\%$	b	1.33675	0.07798	1.51940	1.15410	0.36529
30		θ	1.05116	0.07022	1.22969	0.87264	0.35705
	$\pi_2 = 20\%$	β	3.25770	0.04519	3.56725	2.94814	0.61912
		a	1.45893	0.00307	1.53194	1.38592	0.14602
	$\pi_1 = 20\%$	b	1.35705	0.06119	1.44844	1.26566	0.18278
60		θ	1.07081	0.05450	1.15785	0.98377	0.17408
	$\pi_2 = 20\%$	β	3.25079	0.02697	3.38527	3.11632	0.26895
		a	1.46698	0.00164	1.51277	1.42119	0.09158
	$\pi_1 = 20\%$	b	1.36689	0.05507	1.41991	1.31388	0.10603
100		θ	1.08008	0.04901	1.12983	1.03032	0.09951
	$\pi_2 = 20\%$	β	3.25003	0.02423	3.33172	3.16834	0.16338

Table 4: ML estimates for the parameters and standard errors for the real data sets based on Type II censoring (r=60% and r=10%)

Application I				
$\boldsymbol{\pi}$	$\boldsymbol{n}$	**Parameters**	**Estimates**	**SE**
		a	1.39967	0.00231
$\pi_1 = 20\%$		b	1.49753	0.02385
		θ	2.49801	1.96433E-7
$\pi_2 = 20\%$		β	1.59843	0.01689
	30			
		a	1.29967	0.14725
$\pi_1 = 20\%$		b	1.39753	0.09810
		θ	2.29801	0.00325
$\pi_2 = 70\%$		β	1.39843	0.06172
Application II				
$\boldsymbol{\pi}$	$\boldsymbol{n}$	**Parameters**	**Estimates**	**SE**
		a	0.79922	0.15080
$\pi_1 = 20\%$		b	1.49874	0.30331
		θ	1.10049	0.17775
$\pi_2 = 20\%$		β	2.21192	0.00013
	20			
		a	0.99922	0.39203
$\pi_1 = 20\%$		b	1.59874	0.83178
		θ	1.30049	0.52881
$\pi_2 = 70\%$		β	2.30092	0.51151